SPRING 2025 MATH 540: QUIZ 9

Name:

1. Verify Gauss's Lemma for $(\frac{11}{13})$. (5 points)

Solution. The quadratic residues mod 13 are: 1,3,4, 9, 10, 12, so that 11 is a quadratic non-residue. Thus, $(\frac{11}{13}) = -1$.

We now calculate the products $1 \cdot 11, 2 \cdot 11, \ldots, \frac{13-1}{2} \cdot 11$, mod 13, as they occur in the interval (-6.5, 6.5). Working mod 13 we get $1 \cdot 11 \equiv -2, 2 \cdot 11 \equiv -4, 3 \cdot 11 \equiv -6, 4 \cdot 11 \equiv 5, 5 \cdot 11 \equiv 3, 6 \cdot 11 \equiv 1$. The number of negative values is 3, and thus by Gauss's Lemma, $(\frac{11}{13}) = (-1)^3 = -1$, as before.

2. Use the law of quadratic reciprocity to show that $(\frac{3}{p}) = \begin{cases} 1, \text{ if } p \equiv \pm 1, \text{ mod } 12 \\ -1, \text{ if } p \equiv \pm 5 \text{ mod } 12. \end{cases}$ (5 points)

Solution. We first note that when $p = 3$, $\frac{p-1}{2} = 1$, so in the quadratic reciprocity formula, we will not write the exponent 1. We also have, $-1 \equiv 11$ mod 12 and $5 \equiv 7$ mod 12, so that if $p \equiv -1$ mod 12, then $p \equiv 11$ mod 12, while if $p \equiv -5$ mod 12, then $p \equiv 7$ mod 12. Thus, the four case are:

$p = 12n + 1$: $(\frac{3}{p}) = (-1)^{6n}(\frac{12n+1}{3}) = (\frac{1}{3}) = 1$.

$p = 12n + 11$: $(\frac{3}{p}) = (-1)^{6n+5}(\frac{12n+11}{3}) = -1 \cdot (\frac{2}{3}) = (-1) \cdot (-1) = 1$.

$p = 12n + 5$: $(\frac{3}{p}) = (-1)^{6n+2}(\frac{12n+5}{3}) = 1 \cdot (\frac{2}{3}) = 1 \cdot (-1) = -1$.

$p = 12n + 7$: $(\frac{3}{p}) = (-1)^{6n+3}(\frac{12n+7}{3}) = (-1) \cdot (\frac{1}{3}) = -1 \cdot 1 = -1$.